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Polynomial Lax pairs for the chiral O(3)-field equations and the Landau–Lifshitz equation

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Abstract. New Lax pairs for the chiral O(3)-field equations and for the Landau–Lifshitz equation have been found. In contrast to the already-known pairs, these pairs are polynomial in the spectral parameter λ. We also find a new 4×4 Lax pair in the case of cnoidal waves for the generalized Landau–Lifshitz equation.

1. Introduction

In recent decades considerable attention has been paid to the so-called soliton equations. It is well known that their remarkable properties (see e.g. [1]) are as a result of being able to apply methods of integration based on the inverse scattering problems, and their modifications such as the $\bar{\partial}$ problem. The main point, however, is the possibility of expressing the given evolution equation (or system of such equations) in the so-called Lax form or, equivalently, in the form of the compatibility condition

$$[\partial_x - U, \partial_t - V] = \partial_t U - \partial_x V + [U, V] = 0 \tag{1}$$

of the two linear systems

$$(\partial_x - U)\Psi = 0 \qquad (\partial_t - V)\Psi = 0. \tag{2}$$

Here, the matrix valued functions U and V depend on the spectral parameter λ and on a number of variables $u_1, u_2, \ldots$ usually called potential functions. The evolution equation (or system of equations) is written in terms of the potential functions. We shall deal with evolution equations with one spatial variable $x \in \mathbb{R}$ and as usual t is the time variable.

At the present time there are a number of approaches to the soliton equations based on a variety of theories, for example the spectral theory of operators, the Lie group and the Lie algebra theory, algebraic and differential geometry and others which are difficult to even list. However, we believe that there is one important problem which is open and until now has not been given much attention. The question is whether the results of the main constructions through which the soliton equations are solved, such as the dressing method of Zakharov–Shabat, the Riemann–Hilbert problem or the finite gap integration method, depend or do not depend on the choice of the UV pair, as the constructions themselves strongly rely upon this choice.

It is clear that compatibility condition (1) is expressed through the commutator $[U, V]$ and, therefore, if U and V belong to a certain Lie algebra g, we can write the same compatibility condition in another faithful representation of g and in this way obtain the same evolution equation. It is not evident whether the construction of the exact solutions mentioned above will be compatible with such an alteration of the representation. One of the authors has devoted time to some aspects of this problem in [2]; it seems that there remains far more to be done, even in the most trivial of cases when the pairs differ by the choice of the representation. The problem becomes more complicated if we introduce essentially different UV pairs. It should be mentioned that finding UV pairs is by no means a straightforward process and that, in order to find them, some good fortune is necessary. Thus, the existence of different pairs is quite a rare phenomena and this possibly explains why the problem we mentioned above has not been given much attention up to now.

2. Polynomial 6×6 pairs for the chiral O(3)-field equations and for the Landau–Lifshitz equation

The system of chiral-field equations can be written in the form [3]

$$\begin{cases} u_t + u_x - u \times Jv = 0 \\ v_t - v_x - v \times Ju = 0 \end{cases} \qquad u^2 = v^2 = 1 \qquad J = \mathrm{diag}(j_1, j_2, j_3) \tag{3}$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ are the two vector functions depending on x and t, and $\times$ denotes the vector product.

The Landau–Lifshitz equation [4]

$$u_t = u \times u_{xx} + u \times Ku \qquad u^2 = 1 \qquad K = \mathrm{diag}(k_1, k_2, k_3) \tag{4}$$

is written in terms of the one vector function $u = (u_1, u_2, u_3)$, where the matrix K plays the same role as J.

For convenience we shall write equations (3) and (4) in a different form. Let us introduce the linear mapping $\mathcal{M} : \mathbb{R}^3 \to \mathrm{so}(3)$ where so(3) is the Lie algebra of the 3×3 skew-symmetric matrices

$$\mathcal{M}(u) = \mathcal{M}(u_1, u_2, u_3) = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}. \tag{5}$$

Clearly,

$$[\mathcal{M}(u), \mathcal{M}(v)] = -\mathcal{M}(u \times v) \tag{6}$$

and, therefore, we can express (3) and (4) in the following equivalent forms.

(A) Chiral-field equations (CFE):

$$\begin{cases} \mathcal{M}(u)_t + \mathcal{M}(u)_x + [\mathcal{M}(u), \mathcal{M}(Jv)] = 0 \\ \mathcal{M}(v)_t - \mathcal{M}(v)_x + [\mathcal{M}(v), \mathcal{M}(Ju)] = 0 \end{cases} \qquad u^2 = v^2 = 1. \tag{7}$$

(B) Landau–Lifshitz equation (LLE):

$$\mathcal{M}(u)_t + [\mathcal{M}(u), \mathcal{M}(u)_{xx}] + [\mathcal{M}(u), \mathcal{M}(Ku)] = 0 \qquad u^2 = 1. \tag{8}$$

Our inspiration to construct 6×6 pairs came from reading the last lines in [5], where the author claimed that there exist UV pairs which are linear in λ for CFE and LLE and gave formulae for the U matrices in both cases. However, as far as we know the final answer has not been obtained there or elsewhere. Moreover, we failed to obtain a pair linear in λ for LLE; the pair we obtained depends quadratically on λ and is, in this sense, completely new.

Below we shall write the 6×6 matrices in the block form $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, where a, b, c, d are 3×3 matrices. Using this notation we present the following UV pairs.

(A) The pair for the chiral O(3)-field equations:

$$
\begin{aligned}
U &= \frac{\lambda}{2}\begin{pmatrix} -\mathcal{M}(u) & 0\\ 0 & \mathcal{M}(v)\end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\mathcal{M}(Jv) & 0\\ 0 & \mathcal{M}(Ju)\end{pmatrix}\\
&\quad + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} -\mathcal{M}(u) & 0\\ 0 & \mathcal{M}(v)\end{pmatrix}\\
V &= \frac{\lambda}{2}\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & \mathcal{M}(v)\end{pmatrix} - \frac{1}{2}\begin{pmatrix} \mathcal{M}(Jv) & 0\\ 0 & \mathcal{M}(Ju)\end{pmatrix} + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & \mathcal{M}(v)\end{pmatrix}
\end{aligned}
\tag{9}
$$

where $\hat{J}=\begin{pmatrix} 0 & J\\ J & 0\end{pmatrix}$ and $\mathrm{ad}_{\hat{J}}A$ represents $[\hat{J}, A]$.

(B) The pair for the Landau–Lifshitz equation:

$$
\begin{aligned}
U &= \frac{\lambda}{2}\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0\\ 0 & -\mathcal{M}(Ju)\end{pmatrix} + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & 0\end{pmatrix}\\
V &= \frac{\lambda^2}{4}\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & 0\end{pmatrix} + \frac{\lambda}{2}\left\{\begin{pmatrix} \mathcal{M}(u\times u_x) & 0\\ 0 & 0\end{pmatrix} + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & 0\end{pmatrix}\right\}\\
&\quad -\frac{1}{4}\begin{pmatrix} \mathcal{M}(J^2u) & 0\\ 0 & 2\mathcal{M}(J(u\times u_x))\end{pmatrix}\\
&\quad +\frac{1}{4}\mathrm{ad}_{\hat{J}}\begin{pmatrix} 2\mathcal{M}(u\times u_x) & 0\\ 0 & \mathcal{M}(Ju)\end{pmatrix}.
\end{aligned}
\tag{10}
$$

It is clear that these pairs are polynomial, in contrast to the already-known pair of Sklianin–Borovik [6, 7] and Cherednik [8] which are elliptic in λ.

In order to obtain the LLE from this pair, one has to additionally put $k_s=-\frac{1}{4}j_s^2$, $s=1,2,3$. It should be mentioned that for $j_s=0$, $s=1,2,3$ pair (10) becomes equivalent to the pair

$$
\begin{aligned}
U &= \frac{\lambda}{2}\mathcal{M}(u)\\
V &= \frac{\lambda^2}{4}\mathcal{M}(u) + \frac{\lambda}{2}\mathcal{M}(u\times u_x).
\end{aligned}
\tag{11}
$$

The nonlinear evolution equation which corresponds to this pair is the Heisenberg ferromagnet equation (HFE):

$$
u_t = u\times u_{xx}.
$$

Taking into account the well known isomorphism between the algebras $so(3,\mathbb{R})$ and $su(2)$, pair (11) could be written in terms of 2×2 matrices. In this way we obtain the well known pair for HFE [8]. There are other questions which arise from the pair (10). As we have already mentioned, the pair which has been used up to now for LLE is the Sklianin–Borovik pair [6, 7] containing elliptic functions in λ. It is known, however, that the HFE is gauge equivalent to the nonlinear Schrödinger equation (see [8]) and that the first operator in the pair for the HFE is gauge equivalent to the Zakharov–Shabat linear problem. It is natural, then, to ask whether the Landau–Lifshitz equation is gauge equivalent to some Schrödinger-like equation. It is evident that the operator $\partial_x - U$ in the Cherednik pair cannot be gauge equivalent to the Zakharov–Shabat-type linear problem. Since the eigenvalues of $\begin{pmatrix} \mathcal{M}(u) & 0\\ 0 & 0\end{pmatrix}$ do not depend on u it is possible that through (10) one can establish the aforementioned gauge equivalence.

3. Polynomial 4×4 pairs for CFE and LLE

The pairs (9) and (10) have an important property which allows us to write them in terms of 4×4 matrices. They both belong to the Lie algebra $\mathrm{so}_{\mathbb{C}}(3,3)$—the complexification of $\mathrm{so}(3,3)$:

$$\mathrm{so}_{\mathbb{C}}(3,3) = \{\mathcal{A} : \mathcal{A} \in \mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^6); \mathcal{A}^T\varphi + \varphi\mathcal{A} = 0\} \qquad \varphi = -\begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}.$$

If $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belongs to $\mathrm{so}_{\mathbb{C}}(3,3)$, then $\alpha^{\mathrm{T}} = -\alpha$, $\delta^{\mathrm{T}} = -\delta$, $\beta^{\mathrm{T}} = \gamma$ and *vice versa*. A simple similarity transformation establishes the isomorphism between $\mathrm{so}_{\mathbb{C}}(3,3)$ and $\mathrm{so}(6,\mathbb{C})$. Indeed, if we introduce the matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathrm{i}\mathbb{I} & -\mathrm{i}\mathbb{I} \end{pmatrix}$$

then the similarity transformation $A \to T^{-1}AT$ converts $\mathrm{so}_{\mathbb{C}}(3,3)$ into $\mathrm{so}(6,\mathbb{C})$. It is well known that $\mathrm{so}(6,\mathbb{C})$ is isomorphic to $\mathrm{sl}(4,\mathbb{C})$—the algebra of 4×4 traceless matrices. Thus, it is possible to write the pairs (9) and (10) in 4×4 form. However, one needs to construct explicitly, the isomorphism between $\mathrm{so}_{\mathbb{C}}(3,3)$ and $\mathrm{sl}(4,\mathbb{C})$. We could not find the explicit form of this isomorphism, which is of course trivial, in terms of Dynkin diagrams:

For this reason we shall briefly sketch how one can obtain an isomorphic Cartan–Weyl basis for these algebras.

In order to present the Cartan subalgebra with diagonal matrices we shall use another representation of $\mathrm{so}_{\mathbb{C}}(3,3) \sim \mathrm{so}(6,\mathbb{C})$. Let us introduce

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix} = R^{-1}.$$

Then it is easy to see that if $RBR \in \mathrm{so}_{\mathbb{C}}(3,3)$ then $B \in \widetilde{\mathrm{so}(6)}$ and *vice versa*, where

$$\widetilde{\mathrm{so}(6)} = \left\{A : A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^{\mathrm{T}} \end{pmatrix}, \beta^{\mathrm{T}} = -\beta, \gamma^{\mathrm{T}} = -\gamma\right\}.$$

Now the Cartan subalgebra h can be defined as

$$h = \left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}, \xi = \mathrm{diag}(\xi_1, \xi_2, \xi_3)\right\} \subset \widetilde{\mathrm{so}(6)}.$$

We shall represent every element $\left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix}\right)$ of the Cartan subalgebra by the vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. The Killing form of $\widetilde{\mathrm{so}(6)}$ is well known:

$$B(X,Y) \equiv \mathrm{tr}(\mathrm{ad}_X\ \mathrm{ad}_Y) = 4\mathrm{tr}XY \qquad X, Y \in \mathrm{so}(6).$$

Let the elements ϵ_i from the dual space h^* be given by

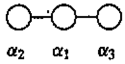
$$\epsilon_i(\xi) = \xi_i \qquad i = 1,2,3.$$

Then it is easy to see that the set of simple roots $\alpha_1, \alpha_3, \alpha_3$ is given by

$$\alpha_1 = \epsilon_1 - \epsilon_2 \qquad \alpha_2 = \epsilon_2 - \epsilon_3 \qquad \alpha_3 = \epsilon_2 + \epsilon_3$$

and the set of all roots is then

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+\alpha_3), \pm(\alpha_1+\alpha_2+\alpha_3)\}.$$

The Dynkin diagram is clearly  since the elements corresponding to α_i through the isomorphism established by R are

$$H_{\alpha_1} = \tfrac{1}{8}(1, -1, 0) \qquad H_{\alpha_2} = \tfrac{1}{8}(0, 1, -1) \qquad H_{\alpha_3} = \tfrac{1}{8}(0, 1, 1).$$

It is then easy to calculate that

$$\langle\alpha_3, \alpha_3\rangle = \tfrac{1}{4} \qquad \langle\alpha_1, \alpha_2\rangle = -\tfrac{1}{8} \qquad \langle\alpha_1, \alpha_3\rangle = -\tfrac{1}{8} \qquad \langle\alpha_2, \alpha_3\rangle = 0.$$

(Here and below we use the notation and normalizations of [9] which are universally accepted.)

Of course, the Dynkin diagram is isomorphic to the diagram of the algebra $\mathrm{sl}(4, \mathbb{C}) \sim \mathrm{A}_3$. As is well known, for this algebra

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y) = 8\mathrm{tr}XY \qquad X, Y \in \mathrm{sl}(4, \mathbb{C}).$$

The Cartan subalgebra is

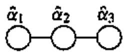
$$\hat{h} = \left\{\mathrm{diag}(h_1, h_2, h_3, h_4), \sum_{i=1}^{4} h_i = 0\right\}.$$

If we introduce $\hat{\epsilon}_i \in \hat{h}^*$ by

$$\hat{\epsilon}_i\,(\mathrm{diag}(h_1, h_2, h_3, h_4)) = h_i$$

then the set of roots $\hat{\Delta}$ is $\hat{\Delta} = \{\hat{\epsilon}_i - \hat{\epsilon}_j,\ i \neq j\}$, $i, j = 1, 2, 3, 4$ and the set of simple roots is given by

$$\hat{\alpha}_1 = \hat{\epsilon}_1 - \hat{\epsilon}_2 \qquad \hat{\alpha}_2 = \hat{\epsilon}_2 - \hat{\epsilon}_3 \qquad \hat{\alpha}_3 = \hat{\epsilon}_3 - \hat{\epsilon}_4.$$

The Dynkin diagram is . We can then define an isomorphism ψ of the root systems which also supports that for the simple roots:

$$\psi(\alpha_1) = \hat{\alpha}_2 \qquad \psi(\alpha_2) = \hat{\alpha}_1 \qquad \psi(\alpha_3) = \hat{\alpha}_3$$

and we can extend ψ by linearity. It is well known that the mapping ψ then generates the isomorphism of the algebras. We shall denote it by Ψ. Finally, we have

$$\begin{aligned} \Psi(E_\alpha) &= \hat{E}_{\psi(\alpha)} \qquad && \alpha \in \Delta \\ \Psi(H_{\alpha_i}) &= H_{\psi(\alpha_i)} \qquad && i = 1, 2, 3 \end{aligned}$$

where $\{E_\alpha, H_{\alpha_i}, \alpha \in \Delta,\ i = 1, 2, 3\}$ and $\{\hat{E}_{\hat{\alpha}}, H_{\hat{\alpha}_i}, \hat{\alpha} \in \hat{\Delta},\ i = 1, 2, 3\}$ are the Cartan–Weyl basis for $\mathrm{so}_{\mathbb{C}}(3, 3) \sim \mathrm{so}(6, \mathbb{C})$ and $\mathrm{sl}(4, \mathbb{C})$, respectively. Using the explicit expressions for the bases given in [9] we can construct the needed isomorphism. We shall put aside the cumbersome but straightforward calculations and present only the final results.

(i) UV pair for the chiral O(3)-field equations:

$$\begin{aligned} U &= \tfrac{1}{2}(A_1 - A_2)(\lambda + \widetilde{J}) \\ V &= \tfrac{-1}{2}(A_1 + A_2)(\lambda + \widetilde{J}) \end{aligned} \tag{12}$$

where

$$A_1 = \frac{1}{2}\begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \tag{13}$$

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix} \tag{14}$$

$$\widetilde{J} = \begin{pmatrix} -j_1 - j_2 + j_3 & 0 & 0 & 0 \\ 0 & -j_1 + j_2 - j_3 & 0 & 0 \\ 0 & 0 & j_1 - j_2 - j_3 & 0 \\ 0 & 0 & 0 & j_1 + j_2 + j_3 \end{pmatrix} \tag{15}$$

(ii) UV pair for the Landau–Lifshitz equations:

$$\begin{aligned} U &= \tfrac{1}{2}A_1(\lambda + \widetilde{J}) \\ V &= \tfrac{1}{2}(\tfrac{1}{2}\lambda A_1 - [A_1, A_{1x}] + \tfrac{1}{2}A_{2J})(\lambda + \widetilde{J}) \end{aligned} \tag{16}$$

where

$$A_{2J} = \frac{1}{2}\begin{pmatrix} 0 & j_1u_1 & j_2u_2 & -j_3u_3 \\ -j_1u_1 & 0 & j_3u_3 & j_2u_2 \\ -j_2u_2 & -j_3u_3 & 0 & -j_1u_1 \\ j_3u_3 & -j_2u_2 & j_1u_1 & 0 \end{pmatrix} \tag{17}$$

$\widetilde{J}$ is given by (15) and A_1 by (13).

We shall make only one comment about the pair for LLE. When $j_s = 0$, $s = 1, 2, 3$ the matrices in this pair become elements of the subalgebra $so(4) \subset sl(4)$. As is well known, $so(4)$ is isomorphic to $so(3) \times so(3)$. Then, for $j_s = 0$ the pair is equivalent to the pair (11) or to the well known 2×2 pair for HFE.

4. A new 4×4 pair for the generalized LLE in the case of cnoidal waves

The simplest generalizations of LLE has the form

$$\begin{cases} u_t = u \times u_{xx} + u \times Jv & J = \mathrm{diag}(j_1, j_2, j_3). \\ v_t = v \times v_{xx} + v \times Ju \end{cases} \tag{18}$$

This describes the anisotropic interaction of two isotropic ferromagnetic lattices. The Lax pair for this system was not known until now, except for the case $u(x, t) = u(x - at)$, $v(x, t) = v(x - at)$ referred to as the case of cnoidal waves. For this case the Lax pair of the classic type

$$\dot{L} = [L, A] \tag{19}$$

was found in [10] with 6×6 matrices belonging to $so(3, 3)$. Using the same isomorphism as in the previous sections, we rewrite this Lax pair in a new form in terms of 4×4 matrices belonging to $sl(4, \mathbb{C})$.

The result of these straightforward calculations can be represented in its final form. The Lax pair for the generalized LLE is

$$\begin{aligned} L &= -2\lambda^2 A_1 A_2 + \lambda\left(a(A_1 + A_2) + [(A_1 + A_2), (A_1 + A_2)_\xi]\right) - \widetilde{J} \\ A &= -2\lambda A_1 A_2 \end{aligned}$$

where $\xi = x - at$ and the operators A_1, A_2 and $\widetilde{J}$ have the same form as in (13)–(15).

We hope that this pair, which looks much simpler than the 6×6 pair, will be helpful for finding new classes of solutions and for stimulating the search for new Lax pairs in general.

5. Conclusions

We propose new 6×6 Lax pairs for the O(3) chiral-field equations and for the Landau–Lifshitz equation in which the dependence on the spectral parameter is polynomial in contrast to the already known pairs in which this dependence is through elliptic functions. Using the classical isomorphism between the algebras sl(4) and so(6) we express these pairs in a much simpler 4×4 form. The same isomorphism is also applied to obtain the pair for the so-called generalized Landau–Lifshitz system in the cnoidal wave case in 4×4 form.

We believe that the new pairs we have obtained will stimulate the discussion whether one obtains the same solutions using essentially different Lax pairs and in finding new Lax pairs especially for the generalized Landau–Lifshitz system. We also suppose that the new pair for the LLE will be useful in finding the equation which is gauge equivalent to the LLE in the same manner as the HFE is gauge equivalent to the nonlinear Schrödinger equation.

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